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Subcritical behavior in the alternating supercritical Domany-Kinzel dynamics

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Abstract. Cellular automata are widely used to model real-world dynamics. We show using the Domany-Kinzel probabilistic cellular automata that alternating two supercritical dynamics can result in subcritical dynamics in which the population dies out. The analysis of the original and reduced models reveals generality of this paradoxical behavior, which suggests that autonomous or man-made periodic or random environmental changes can cause extinction in otherwise safe population dynamics. Our model also realizes another scenario for the Parrondo's paradox to occur, namely, spatial extensions.

PACS. 02.50.Ga Markov processes – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 87.23.Cc Population dynamics and ecological pattern formation

1 Introduction

Ecological and sociological dynamics are often described by systems of locally interacting agents. Cellular automata are broadly used for modeling such dynamics to characterize, for example, survival probability, percolation, and critical phenomena, which are relevant to real situations [1]. Among the class of probabilistic cellular automata is the Domany-Kinzel (DK) model, which is a two parameter family of Markov processes on a one-dimensional lattice with discrete time [2,3]. In this paper, we report a counterintuitive phenomenon of the DK model: particles eventually die out when two supercritical DK dynamics alternate with some appropriate orders. This behavior is robust against parameter changes. We also analyze the reduced dynamics such as the pair approximation and a canonical model to guarantee that this phenomenon is preserved in much simpler models. As a generalization, dynamic environmental changes can extinguish a population even if the snapshot dynamics is supercritical at any given moment. These alternating DK dynamics also realize a new scenario for the Parrondo's paradox [4–6] to occur, that is, introduction of the space.

2 DK model

In the DK model [2,3], each site either accompanies a particle (denoted by •) or is empty (denoted by ∘) at any instant. The space can be identified with a subset of the set of integers $\mathbf{Z}$, and let $\xi_n \subset \mathbf{Z}$ be the set of the sites that have particles at discrete time $n \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. The stochastic evolution rule at each site $x \in \mathbf{Z}$ is independently described by $P(x \in \xi_{n+1}|\xi_n) = f(|\xi_n \cap \{x-1, x+1\}|)$ where $f(0) = 0$, $f(1) = p_1$, $f(2) = p_2$, and $(p_1, p_2) \in [0, 1]^2$. In other words, the probability that a particle emerges is determined by the number of the particles in the nearest neighborhood in the previous time, as shown in Figure 1. Each realization of the spatiotemporal process is expressed in the form of a configuration $\xi \in \{0, 1\}^{\mathbf{S}} = X$ with $\mathbf{S} = \{s = (x, n) \in \mathbf{Z} \times \mathbf{Z}_+ : x + n = \text{even}\}$. The region of the supercritical parameter sets (p_1, p_2) for which particles survive for infinite time with positive probability can be numerically obtained, and it occupies an upper-right

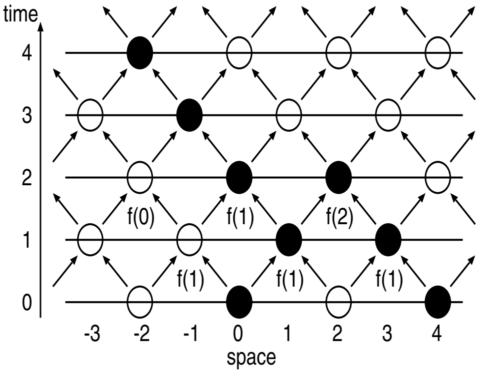

Fig. 1. Schematic diagram showing the DK probabilistic cellular automaton.

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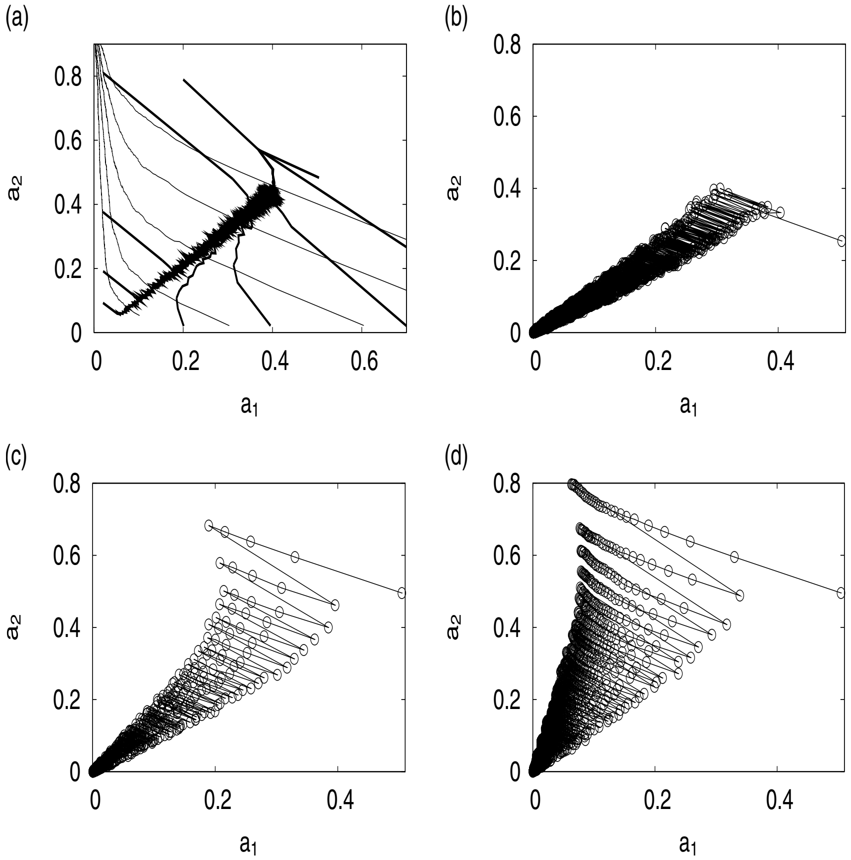

Fig. 2. (a) Trajectories of the DK model for dynamics A, with $(p_1, p_2) = (0.52, 1)$ (thin lines) and those for dynamics B with $(p_1, p_2) = (0.76, 0.76)$ (thick lines). The other panels show population dynamics when we repeat (b) AB, (c) A^4B, and (d) $A^{30}B$. The initial conditions for (b, c, d) are $(a_1(0), a_2(0)) = (0.5, 0.25)$.

area in the p_1-p_2 space [2,3,7]. The DK model is equivalent to the directed bond percolation model on a square lattice when $(p_1, p_2) = (p, 2p - p^2)$ and to the directed site percolation model when $p_1 = p_2 = p$ [2,3,7]. Another special case is Wolfram's rule 90 deterministic cellular automaton [1] which is realized with $(p_1, p_2) = (1, 0)$. The simplicity of the DK model enables us to investigate interesting properties from the viewpoint of statistical physics and applications, such as quasistationary particle density [8–12], critical phenomena and phase transitions [2,3,8–14], survival probabilities [15], and duality [16–18].

Let us denote by $P_n(\cdot)$ the probability that an event occurs at time n. Here an event means a state of consecutive sites, or a sequence of • and ∘. For clarity, we often plot trajectories in the two-dimensional space spanned by the order parameters defined with $a_2(n) \equiv P_n(\bullet\bullet)$ and $a_1(n) \equiv P_n(\bullet\circ) + P_n(\circ\bullet)$. With $a_0(n) \equiv P_n(\circ\circ)$, it follows that $a_1(n) \geq 0$, $a_2(n) \geq 0$, and $a_1(n) + a_2(n) = 1 - a_0(n) \leq 1$. The origin $(a_1, a_2) = (0, 0)$ is an absorbing fixed point corresponding to the population death. In the following numerical simulations, the lattice size is 10 000, and the periodic boundary conditions are assumed.

With some initial conditions, trajectories of the DK model are shown in Figure 2a for $(p_1, p_2) = (0.52, 1)$ (thin lines) and for (0.76, 0.76) (thick lines). The DK dynamics corresponding to these parameter sets are termed dynamics A and dynamics B, respectively. When $p_2 = 1$, particles emerge or die only at kinks where • and ∘ face each other. In this case, the dynamics of kinks are identical to the coalescing random walk, and the entire space is eventually occupied by particles with a positive probability if and only if $p_1 > 0.5$ [7]. Therefore, dynamics A is supercritical. On the other hand, the DK model with $p_1 = p_2$ is equivalent to the directed site percolation. Restricted onto this line, $p_1 = p_2 = 0.75$ is a mathematically rigorous upper bound for the subcritical regime [19], whereas the critical value is numerically estimated to be about $p_1 = p_2 = 0.7055$ [20,21]. Because of the attractiveness $(p_1 \leq p_2)$, the natural intuition that more particles are likely to survive with larger p_1 and p_2 actually holds [7,15]. Therefore, dynamics B is also supercritical. Accordingly, trajectories of dynamics A converge to the all • state, and those of B converge to the stochastic stable fixed point $(a_1, a_2) \cong (0.39, 0.42)$.

3 Population death in alternating DK dynamics

Next, we alternatively apply A and B. A typical trajectory is shown in Figure 2b with the Bernoulli initial distribution with density 0.5, which yields $(a_1(0), a_2(0)) = (0.5, 0.25)$. Surprisingly, particles eventually die out. This behavior is not sensitive to the choice of initial conditions.

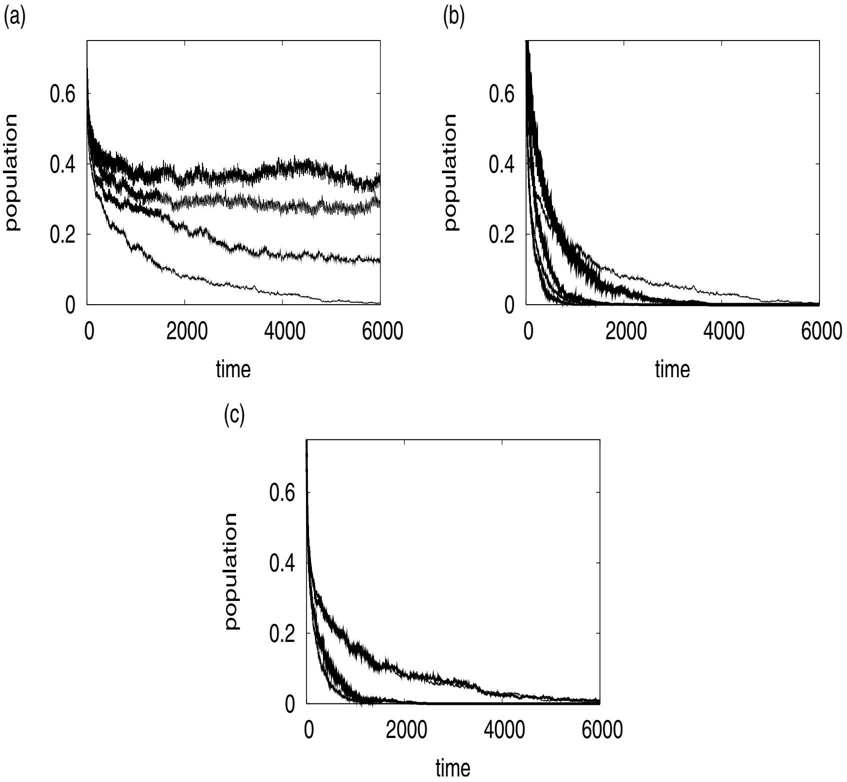

Fig. 3. Dynamics of the population size when we repeat (a) A^kB^k with $k = 1$ (thinnest line), 2, 3, and 4 (thickest line), (b) A^kB with $k = 1$ (thinnest), 2, 4, 15, 30 (thickest), and (c) AB^k with $k = 1$ (thinnest), 2, 3 and 4 (thickest). In (b), the lowermost line corresponds to $k = 4$. In (c), the upper lines, which are nearly superimposed, correspond to $k = 1$ and 3, whereas the lower lines correspond to $k = 2$ and 4.

It also persists against changes in p_1 or p_2 as far as the individual dynamics are not extremely supercritical and the stable stochastic fixed points for the two systems are separated enough. Especially, extensive numerical simulations suggest that this population death is enhanced when one of the component dynamics is nonattractive, or $p_1 > p_2$.

An important cause for the population death is how the trajectories of dynamics A and those of dynamics B cross. As shown in Figure 2a, if a state in the a_1-a_2 space evolves along a trajectory of dynamics A, in terms of dynamics B, the state gradually slides down to trajectories associated with initial conditions with fewer particles. In other words, from the viewpoint of dynamics A (resp. B), the population once decreases under dynamics B (resp. A) before it revives and reaches the nontrivial stable fixed point. Therefore, by switching the dynamics between A to B before the population effectively starts to grow, the number of the particles gradually decrease to zero. Survival results if A or B is applied long enough before switching to the other. To demonstrate this, we confine ourselves to the cases in which a block of k A's and k B's are alternatively applied, which we denote by A^kB^k. As shown in Figure 3a, the population is more likely to survive as k increases.

The population death by alternation is an example of the Parrondo's game in which a combination of two losing (winning) stochastic games can counterintuitively end up with a winning (losing) game [4–6,22]. In this context, the results in Figure 3a agree with those for the original Parrondo's game in which the paradoxical effect becomes small as k is raised [6]. A more general concern is how the arrangement of A and B affects the upshot. Since it appears quite difficult to derive the optimal ordering of A and B among all the possible sequences [6,22], we only deal with some representative cases.

The population dynamics when a chain of A is periodically punctuated by just one B, which is denoted by A^kB, are shown in Figure 3b. This figure together with additional numerical simulations suggests that the paradoxical effect is most manifested, or the population dies out most rapidly, with $k = 4$. This is presumably because dynamics B correspond to the critical line of the attractiveness ($p_1 = p_2$). For this reason, in an upper-left region in the a_1-a_2 space, an application of the near-nonattractive B kills more particles when there exist more particles. This view is supported by Figures 2b and 2c in which we compare the dynamics with AB and those with A^4B. Then, the convergence to $(a_1, a_2) = (0, 0)$ is accelerated by a larger k in a small k regime. However, with a much larger k, the number of particles changes little for most of the time (Fig. 2d). In this regime, the population death is slowed down as k increases.

For sequences in the form AB^k, the parity effect is manifested. As shown in Figure 3c, the population death

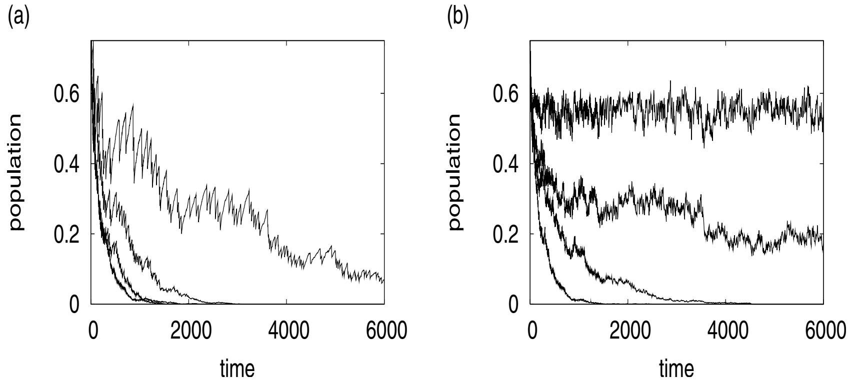

Fig. 4. Population dynamics when A and B randomly appear with probability r and $1-r$, respectively. (a) $r = 0.02$, 0.05, 0.1, 0.2 (from upper to lower lines), and (b) $r = 0.2$, 0.4, 0.5, 0.75 (from lower to upper lines).

is faster when k is even. This is again because dynamics B is nearly nonattractive. As is prominent in nonattractive DK dynamics, the motion in the a_1-a_2 space under dynamics B is somewhat sensitive to the current state. More specifically, simple repetition of B yields a damped oscillation in the early stage. Therefore, if the initial state is located in a upper-left region, the number of particles drops more when B is repeated even times before being interrupted by one A.

The random arrangement of A and B is also of interest [4,6,22] because real environments can be random rather than perfectly periodic. To mimic simple random environments, we choose A and B independently at each time step with probability r and $1-r$, respectively. Obviously, the population death does not occur with $r = 0$ or with $r = 1$, which prescribes the sequence purely of B and that of A, respectively. Figure 4 and the extensive parameter search reveal that the paradoxical effect is maximized when $r \cong 0.2$. This value coincides with the optimal mixing ratio for the family of deterministic sequences investigated above, namely, A^4B.

It is also essential for the paradox that population change rates are proportional to the population size as shown in Figure 5 (crosses). Owing to this property, the size of the population exponentially shrinks to a very small level (Figs. 2–4). Then, particles become extinct in finite time because of stochasticity and the absolute stability of the fixed point $(a_1, a_2) = (0, 0)$. If change rates are too high even for minute population mass, a trajectory that happens to have approached the origin more likely escape the vicinity of the origin to avoid the population deaths.

4 Pair approximation

To take a closer look at the paradox, we analyze the deterministic dynamics derived by the pair approximation of the DK model, which we call the PA dynamics [9–12]. In the pair approximation, any events at two sites separated by a distance more than one are supposed to be independent of each other. For example, $P_n(\bullet|\circ\bullet) = P_n(\bullet\circ\bullet)/P_n(\circ\bullet)$ is approximated by $P_n(\bullet|\circ) = P_n(\bullet\circ)/P_n(\circ)$, where $P_n(\cdot|\cdot)$ denotes the conditional probability. Accordingly, probabilities of any events involving three or more consecutive sites are decomposed into one- or two-site probabilities. With this approximation, the two-dimensional PA dynamics are written as follows:

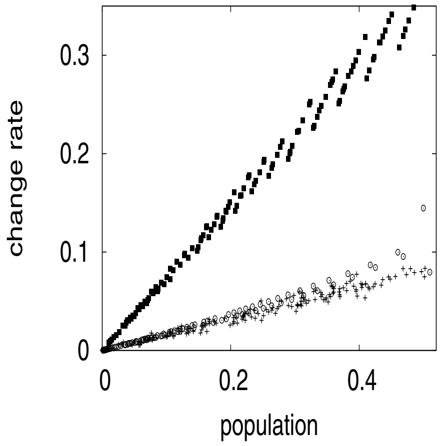

Fig. 5. Population change rates in terms of the population size for the DK (crosses), PA (circles), and canonical (squares) dynamics. The change rates for the DK and PA dynamics are measured by the Euclidean distances of two points with unit time difference in the a_1–a_2 space. The population size is equal to $a_1(t)/2 + a_2(t)$ for the DK and PA dynamics and is defined to be $\sqrt{x^2 + y^2}$ for the canonical dynamics.

$$\begin{aligned} a_2(n+1) &= p_2^2 P_n(\bullet\bullet\bullet) + p_1 p_2\{P_n(\bullet\bullet\circ) + P_n(\circ\bullet\bullet)\} \\ &\quad + p_1^2\{P_n(\bullet\circ\bullet) + P_n(\circ\bullet\circ)\} \\ &\cong \frac{(2p_2a_2(n) + p_1a_1(n))^2}{4b_1(n)} + \frac{p_1^2 a_1(n)^2}{4b_0(n)}, \end{aligned} \quad (1)$$

$$\begin{aligned} a_1(n+1) &\cong p_2(1-p_2)\frac{2a_2(n)^2}{b_1(n)} \\ &\quad + (p_1 + p_2 - 2p_1p_2)\frac{a_1(n)a_2(n)}{b_1(n)} \\ &\quad + p_1(1-p_1)\frac{a_1(n)^2}{2b_0(n)b_1(n)} + p_1\frac{a_1(n)a_0(n)}{b_0(n)}, \end{aligned} \quad (2)$$

where $b_1(n) = a_2(n) + a_1(n)/2$, $b_0(n) = 1 - b_1(n)$ and $a_0(n) = 1 - a_1(n) - a_2(n)$. Trajectories of the PA dynamics are shown in Figure 6a for two sets of supercritical

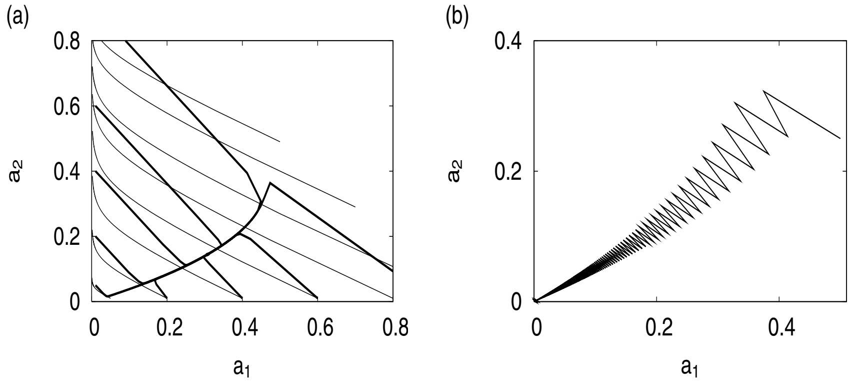

Fig. 6. (a) Trajectories of the PA dynamics for $(p_1, p_2) = (0.52, 1)$ (thin lines) and those for $(p_1, p_2) = (0.66, 0.66)$ (thick lines). (b) Population dynamics in the alternating PA dynamics starting from $(a_1(0), a_2(0)) = (0.5, 0.25)$.

parameter sets: $(p_1, p_2) = (0.52, 1)$ (thin lines) and (0.66, 0.66) (thick lines). In accordance with Figure 2a, the individual PA dynamics own stable fixed points near (0, 1) and (0.308, 0.145). However, as shown in Figure 6b, the population dies out when they alternate. Although the supercritical parameter region of the PA dynamics deviates from that of the DK counterparts, the results for the PA dynamics qualitatively agree with those for the DK dynamics shown in Figure 2.

The Parrondo's paradox is unlikely to happen in one-dimensional systems since they lack auxiliary dimensions that counteract the seeming tendency of population increase. To demonstrate this, let us imagine the simplistic mean-field approximation in which a joint probability is approximated by a product of single-site probabilities (e.g. $P_n(\circ\bullet) \cong P_n(\circ)P_n(\bullet)$). The approximate one-dimensional sytem is written as

$$b_1(n+1) = p_2 b_1(n)^2 + 2p_1 b_1(n)(1 - b_1(n)), \qquad (3)$$

which has fixed points $b_1 = 0$ and $b_1 = (2p_1 - 1)/(2p_1 - p_2)$ [9–12]. Let us pick two mean-field dynamical systems so that their nontrivial fixed points are positive and stable, with $2p_1 > p_2$ and $p_1 > 1/2$ satisfied. Then, when two mean-field dynamics alternate, the particle density $b_1(n)$ just moves between these two fixed points in the long run. Accordingly, the population never dies out, and no paradoxical phenomenon occurs.

5 Canonical model

To generalize the Parrondo's paradox found for the DK and PA dynamics, we construct a simple canonical model with dimension two, which is the presumed minimal degree of freedom for the paradox. As we have mentioned, the relevant features of the DK and PA dynamics can be summarized as follows.

(i) Trajectories of dynamics A and those of dynamics B transverse in the way as shown in Figures 2a and 6a. More specifically, in the a_1-a_2 space, the slope of a trajectory of dynamics A (thin lines) is less negative than that of a trajectory of dynamics B (thick lines) at the crossing point, at least in a certain region.

(ii) Each of A and B is not applied too many times successively. In other words, k in the sequence A^kB^k, A^kB, or AB^k should be small enough, as explained with Figures 2d and 3a.

(iii) Population change rates are proportional to the population size. To weaken the condition may result in the same conclusion just with a different convergence rate. Here we assume this linearity for our canonical model.

The paradox also relies on the following implicit assumptions.

(iv) Dynamics A and dynamics B have sufficiently separated nontrivial fixed points.

(v) The origin is the deterministically unstable but stochastically reachable fixed point for both A and B.

Figures 2a and 6a further indicate that the origin and the nontrivial fixed point are connected by applying B infinitely many times (thick lines). However, it is not true for A (thin lines) because A is nongeneric in the sense that all the points on the a_2 axis are fixed points. Actually, no point with $a_1 = 0$ and $0 < a_2 < 1$ is realizable because it would mean that two consecutive sites take state ●● and ○○ with positive probabilities but not ●○ or ○●. The total size of the boundaries between clusters of ● and those of ○, or $a_1(n)$, determines the population change rates [7]. It declines to zero as a point in the a_1-a_2 space approaches the a_2 axis.

In fact, we have chosen (p_1, p_2)=(0.52, 1) for dynamics A just because the obtained DK dynamics is rigorously supercritical. The paradoxical dynamics appear robustly against changes in p_1 and p_2, which can make the dynamics generic. With this in mind, we construct a two-dimensional continuous-time system that satisfies the conditions listed above. We propose to alternate two dynamical systems:

dynamics A

$$\dot{x} = -x,$$
$$\dot{y} = -\lambda y(1 - y),$$

and

dynamics B

$$\dot{x} = -\lambda x(1 - x),$$
$$\dot{y} = -y,$$

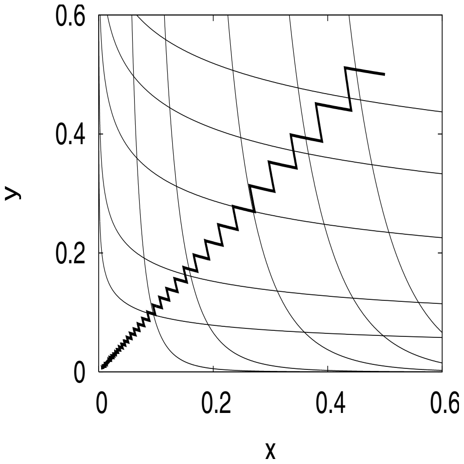

Fig. 7. Trajectories of dynamics A and those of dynamics B of the canonical model (thin lines), superimposed by those of the alternating dynamics starting from $(x, y) = (0.5, 0.5)$ (thick line).

where $0 < x, y < 1$. The properties (iii), (iv), and (v) are obviously satisfied, with (iii) also supported by Figure 5 (squares). Both dynamics A and B have a saddle at the origin. The point $(0, 1)$ of A and $(1, 0)$ of B are stable equilibria, and each of them is connected to the origin by a heteroclinic orbit. The property (i) is satisfied if $0 < \lambda < 1$. To guarantee (ii), we set $\lambda = 0.3$ and the duration of each dynamics equal to 0.15. Figure 7 summarizes flows of the individual dynamics (thin lines) and those of the alternating dynamics (thick lines). We again observe the paradox that the alternating dynamics lead the state toward the origin.

6 Conclusions

We have shown using the DK model and its simplifications that mixtures of two supercritical dynamics can yield subcritical dynamics in which the population dies out. This counterintuitive behavior occurs if individual component dynamics have at least dimension two and satisfy certain criteria. The property (i) is characteristic of the DK or the canonical model, and it agrees with some natural occasions but not with others [23]. The other four requirements do not seem to spoil the reality. The properties (iii) and (v) are satisfied when production rates are primarily proportional to the population mass, which is quite common for ecological and social systems [23]. Periodical and random environmental changes comply with (ii) and (iv). Such changes may be also caused by continual, periodic, or random human control of a system with the aim of moving the stable fixed point to more desirable one. However, our results indicate that environmental changes or oddly managed control measures can cause a total disaster even if the system instantaneously stays in a supercritical 'good' regime all the time. The other way round, there is a general expect that a situation that is subcritical at any moment can be changed into a supercritical one with appropriate controls, which is originally illuminated by the Parrondo's paradox [4–6,22]. In the context of the Parrondo's paradox, our model provides another mechanism of its occurrence in addition to inhomogeneous game rules or players with memory [5], namely, spatial extension.

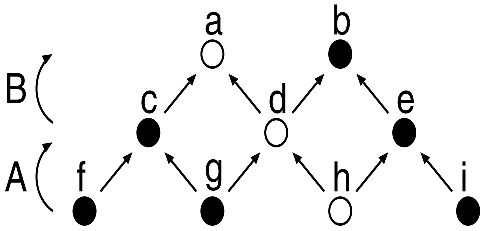

Fig. 8. Alternating dynamics with sequence AB compared with standard 3-neighborhood PCA.

Lastly, we can regard a block of sequence of A and B, such as A^kB and AB^k, as a transformation done in just one step. By doing so, the alternating DK model seems similar to m-neighborhood probabilistic cellular automata (PCA) with $m \geq 3$. For PCA, phase diagrams have been studied in simple cases where the dynamical rule depends only on the number of particles in the neighborhood with $m = 3$ [13,14]. However, the model proposed here is more complex even with the simplest sequence $ABAB\ldots$, which should be compared to PCA with $m = 3$. One reason is that outcomes depend not only on the number but also on the arrangements of particles in a neighborhood [8]. For instance, it is easy to verify $P_n(\bullet|\bullet\circ\bullet) \neq P_n(\bullet|\bullet\bullet\circ)$. More importantly, the alternating DK dynamics are not special cases of finite-range PCA. To illustrate this, let us consider $ABAB\ldots$ In Figure 8, the state of site a, which we write $\xi(a)$ depends on $\xi(f)$, $\xi(g)$, and $\xi(h)$, while $\xi(g)$, $\xi(h)$, and $\xi(i)$ put together determine $\xi(b)$. In 3-neighborhood PCA, there exists no intermediate layer of sites such as c, d, and e. Therefore, once $\xi(f)$, $\xi(g)$, $\xi(h)$, and $\xi(i)$ are given, $\xi(a)$ and $\xi(b)$ are independant. On the other hand, in our model, $\xi(a)$ and $\xi(b)$ are partially correlated, or correlated even conditioned by $\xi(f)$, $\xi(g)$, $\xi(h)$, and $\xi(i)$. This is because both $\xi(a)$ and $\xi(b)$ depend on $\xi(d)$. By the same token, the infinite-range correlation is generated just after single application of AB, which prohibits use of powerful duality equations [16–18]. In this sense, our model stipulates a class of infinite particle systems different from ordinary PCA. However, on the analogy of the Parrondo's paradox, the phenomena reported in this paper may hold for PCA and more general alternating dynamics with general neighborhood sizes.

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